How to calculate Colon Ideals in GCD Domains in particula in UFD Domains

Theorem:

Let R be a GCD domain, $I \trianglelefteq R$ and $S \subseteq R$. Moreover let $I =< i_\alpha >_{\alpha \in J}$ and $S = \{s_\beta\}_{\beta \in K}$ for some index sets I, K. Then the following relation holds:

$$\bigcap_{\beta \in K} \left\langle \frac{i_\alpha}{gcd(i_\alpha, s_\beta)} \right\rangle_{\alpha \in J} \subseteq I : S \tag{1}$$

proof: We can take $< S >$ instead S as $I :< S >= I : S$. This enables us to eliminate maybe a few of s_β's and take only $\{s_\beta\}$ that generates $< S >$. As, $I : S = I :< S >= I :< \{s_\beta\} >= I : \{s_\beta\}_{\beta \in K}$ which is sometimes useful to reduce computation.
Coming back to proof, take

$$x \in \bigcap_{\beta \in K} \left\langle \frac{i_\alpha}{gcd(i_\alpha, s_\beta)} \right\rangle_{\alpha \in J}$$

$$\Rightarrow x \in \left\langle \frac{i_\alpha}{gcd(i_\alpha, s_\beta)} \right\rangle_{\alpha \in J}, \forall \beta \in K$$

$$\Rightarrow x = \sum_{\substack{\alpha \in J' \subseteq J \\ J' \text{ is finite}}} r_\alpha \frac{i_\alpha}{gcd(i_\alpha, s_\beta)}, \forall \beta \in K$$

$$\Rightarrow x s_\beta = \sum_{\substack{\alpha \in J' \subseteq J \\ J' \text{ is finite}}} r_\alpha \frac{i_\alpha s_\beta}{gcd(i_\alpha, s_\beta)}, \forall \beta \in K$$

$$\Rightarrow x s_\beta = \sum_{\substack{\alpha \in J' \subseteq J \\ J' \text{ is finite}}} r_\alpha i_\alpha t_{\alpha,\beta}, \forall \beta \in K$$

where, $t_{\alpha,\beta} = \frac{s_\beta}{gcd(i_\alpha, s_\beta)}$ and hence $x s_\beta \in I$. This means $x \in I : \{s_\beta\} = I : S$.

Corollary:

To find the equality in (1) we have to check is there any element outside $\bigcap_{\beta\in K}\left\langle\frac{i_\alpha}{gcd(i_\alpha,s_\beta)}\right\rangle_{\alpha\in J}$. To find element outside this ideal we can look at non-zero element in $\frac{R}{\bigcap_{\beta\in K}\left\langle\frac{i_\alpha}{gcd(i_\alpha,s_\beta)}\right\rangle_{\alpha\in J}}$. For a non-zero element $x+\bigcap_{\beta\in K}\left\langle\frac{i_\alpha}{gcd(i_\alpha,s_\beta)}\right\rangle_{\alpha\in J}$, it is sufficient to check wheter $x\in I:S$ as corresponding nonzero elements in R will be $x+i$ for all $i\in\bigcap_{\beta\in K}\left\langle\frac{i_\alpha}{gcd(i_\alpha,s_\beta)}\right\rangle_{\alpha\in J}$. But if $x+i\in I:S$ so is x. Now if we have such x then we have another ideal $\left\langle x,\bigcap_{\beta\in K}\left\langle\frac{i_\alpha}{gcd(i_\alpha,s_\beta)}\right\rangle_{\alpha\in J}\right\rangle\subseteq I:S$ and repeat the above process. Else, we get the equality.

Example:

Let $R=F[x,y]$, $I=<xy^2,x^2y>$ and $S=\{y^2\}$.
By equation (1) we have,
$\left\langle\frac{xy^2}{gcd(xy^2,y^2)},\frac{x^2y}{gcd(x^2y,y^2)}\right\rangle=<x,x^2>=<x>\subseteq I:S$

Now, find non-zero element in $\frac{F[x,y]}{<x>}\cong F[y]$. Any non zero element of $F[y]$ is of the form $f(y)$ and taking pre-image under th natural isomorphism ($\phi(f(x,y)+<x>)=f(x,y)(mod\ x)$)

So, $\phi^{-1}(f(y))=f(y)+<x>$.
And hence check whether $f(y)\in I:S$?
$f(y)y^2\notin I$ as elements in I must have atleast 1 degree in indeterminate x. Hence $f(y)=0$ and hence
$I:S=<x>$