Manifold Theory

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Motivation

After reading and making this project I am led to the following conclusions:

- Manifold theory is basically a generalisation of euclidean space but rather more realistic than Euclidean Spac.Basically, manifold study comes from the need of studying real life objects and calculus on them to find maximum and minimum of some height function!
  For example, studying contour and landcapes and studying what happens if we passes through some points where every directional derivative vanishes, what changes happen in the topology of the sub-manifold defined by $X_p = f^{-1}(-\infty, f(p))$.
- In this purpose we define calculus on manifold or arbitrary structures of real life which we can visualize as a euclidean space when zoomed in locally. Then differentiation being local property we can relate it to more theoritical aspect the euclidean space and study it easily as mathematics is easy in euclidean spaces. But euclidean space is not realistic. It just came to visualize manifolds or real life objets in the

Definition Of Topological Spaces

most symmetric and easy way so, that we can develop mathematics on it symmetrically then can extend it on Manifolds via approximating them locally to euclidean spaces.

• Let X be a any non-empty set and let $\tau \subseteq \mathcal{P}(\mathrm{X})$. Now if τ has the following properties viz. :

1. $\phi, X \in \tau$.
2. Let $\{U_\alpha\}_{\alpha \in J}$ be any arbitrary collection of elements of τ, then $\cup_{\alpha \in J} U_\alpha \in \tau$.
3. Let $\{U_i\}_{i=1}^{n} \subseteq \tau$ the $\cap_{i=1}^{n} U_i \in \tau$.

if the above three properties are satisfied the (X, τ) is called a topological space.

Definition of Manifold

• Let (X, τ) be any topological space which is hausdorff and second countable and locally euclidean i.e. locally homeomorphic to a open subset of $\mathbb{R}^n$ for a fixed n. That is there is a homeomorhism $\phi_\alpha : U'_\alpha(\subseteq X) \to U_\alpha(\subseteq \mathbb{R}^n)$.

• Charts: For a point $p \in X$ if there is a homomorphism as in the above definition then we say (ϕ_α, U'_α) is a chart around p.

Two charts $(U, \phi), (V, \psi)$ are called compatible iff $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is diffeomorphic. This map is called **transition map!**

Figure of Transition Map

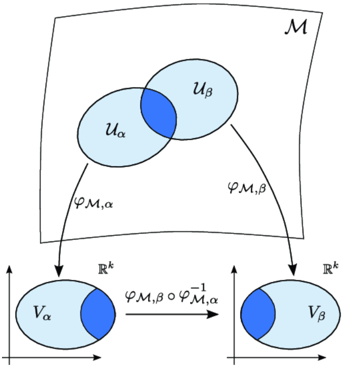


Figure: transition map

Dimesion of Manifold

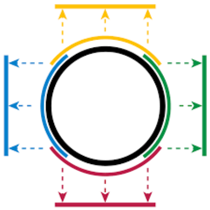
We define local dimension of a manifold at a point $p \in X$ to be the $n \in \mathbb{N}$ such that at that point we get a chart say $\phi : U \to U^{'}(\subseteq \mathbb{R}^n)$.
Is the above definition well defined? I.e. can't we get some other chart viz. $\psi : V \to V^{'} \subseteq \mathbb{R}^k$, where $k \neq n$.
If so, then $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is homeomorphism but as u see $\psi(U \cap V) \subseteq V^{'} \subseteq \mathbb{R}^k$ is open subset of $\mathbb{R}^k$ but $\phi(U \cap V) \subseteq U^{'} \subseteq \mathbb{R}^n$ is a open subset of $\mathbb{R}^n$. But two open subsets of different dimensional euclidean spaces can't be homeomorphic as if they be homeorphic then the whole this boils down to the homeomophism between $\mathbb{R}^n$ and a open subset of $\mathbb{R}^k$ when $k \geq n$ (w.l.o.g.). And this is not possible can be proved in various ways. One of the way is using banachspace idea that any f.d. vector subspace is banach and hence $\mathbb{R}^n$ is closed in $\mathbb{R}^k$. Contradiction of the connectedness of $\mathbb{R}^k$.
We now proved that local dimension is well defined! Now, we moved forward to defining global dimension. For this:-

Lemma 0 :

•**Lemma 0** we restrict ourselves to connected spaces in particular connected manifolds, then then local dimension becomes global dimension. proof: If the manifold is connected then there exits no chart in an atlas not intersecting with other charts otherwise that chart and union of the rest charts will form a seperation! But as we have seen in the above discussion that intersection of two charts will make same local dimension through out the both charts making the local dimension transmitted globally.

•**Atlas:**Collection of pairwise compatible charts whose union is whole X. An atlas is denoted by $\mathcal{A} = \{U_\alpha : \cup_\alpha U_\alpha = X \text{ and } U_\alpha, U_\beta \text{ is compatible}\}$.

There may be many atlases on a particular manifold for example,



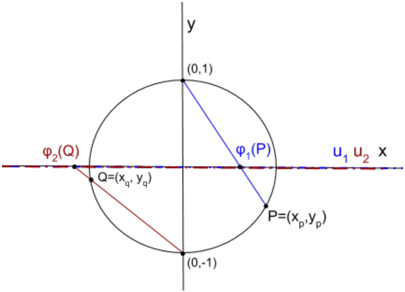


Figure: Vertical Projection Charts(left) and Stereographic projection chart(right)

Hence, we get two atlases viz. $\mathcal{A}_1 = \{yellow, red, green, blue\}$, $\mathrm{A}_2 = \{circle - \{southpole\}, circle - \{northpole\}\}$. But if you calculate, you will eventually find that $\mathcal{A}_1 \cup \mathcal{A}_2$ is again an atlas. Hence we can take only one larger collection viz. the union! But not always all atlas union end upto again an atlas!

Hence, we introduce concept of equivalence classes on collection of all possible atlases on a given manifold X.

• Two atlases $\mathcal{A}_1, \mathcal{A}_2$ is called related iff $\mathcal{A}_1 \cup \mathcal{A}_2$ is again an atlas!

Now, it is easy to verify that this is an equivalence relation of collection of atlases on a given manifold!

•**Lemma1:** The above relation is an equivalence relation!

Proof of Lemma 1 :

proof: reflexivity and symmetricity is trivial. Only non-trivial to check is transitivity!
Let $\mathcal{A}_1 \sim \mathcal{A}_2$ and $\mathcal{A}_2 \sim \mathcal{A}_3$, then we take any $U \in \mathcal{A}_1, W \in \mathcal{A}_3$. Now let $\mathcal{A}_2 = \{V_\alpha\}$. We need to prove U, W is compatible and that completes the proof! Then let $\phi : U \to U'$ and $\psi : V \to V'$. Then,
$\phi \circ \psi^{-1} : \psi(U \cap V_\alpha \cap W) \to \phi(U \cap V_\alpha \cap W)$ can be written as $\phi \circ \zeta_\alpha^{-1} \circ \zeta_\alpha \circ \psi^{-1} = \phi \circ \psi^{-1}\ \forall \alpha$. But $\forall \alpha$, $\phi \circ \zeta_\alpha^{-1}, \zeta_\alpha \circ \psi^{-1}$ are diffeomorphic due to compatibility of $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_2, \mathcal{A}_3$. Hence, $\phi \circ \psi^{-1}$ is diffeomorphic on $U \cap V_\alpha \cap W$ and we have $U \cap W = \cup_\alpha (U \cap V_\alpha \cap W)$. Then by very definition of differentiability we get $\phi \circ \psi^{-1}$ is diffeomorphic on $U \cap W$. Provind the lemma!
Hence, we get a equivalence class of atlases on a manifold! And we call one equivalence class a *Differentiable structure*!

Different Differential structures:

Now, we can have a multipe differentiable structures on a manifold of various dimesions! Below is a chart of differentible stuctures on various dimensional spheres!

Dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Smooth types	1	1	1	≥1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24

Figure: Differentiable structures on spheres

We see clearly that dim≤ 3, we have only one differentiable structure. This is in general true for any manifold with dim less than or equal to 3.
A manifold with a differentiable structure is called a differentiable maifold and if the transition maps are smooth maps(i.e. all order partial derivatives exits).
Now, we move to define our most basic and important setup viz. differentiability of a function from one manifold to other!

Remarks:

Now we can draw some remark from the above discussions viz.

1. If $p \in U \subset X$, then $\phi(p) = (\phi_1(p), \phi_2(p), ..., \phi_n(p)) \in \mathbb{R}^n$.
2. Since ϕ is cont., so $\phi_i : U \to \mathbb{R}$ is a cont. for each $i = 1(1)n$.
3. The pair (U, ϕ) is called a coordinate neighborhood!(or a coordinate chart or a chart) of X.
4. $(\phi_1, \phi_2, ..., \phi_n)$ is called local coordinate system on (U, ϕ).

Example of a Smooth Manifold:

The n-sphere $\mathbb{S}^n = \{x = (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : ||x|| = 1\}$ $\forall n \in \mathbb{N}$ can be endowded with a stereographic atlas on it viz. $\mathcal{A} = \{U_1, U_2\}$ where $U_1 = \mathbb{S}^n - \{(0, 0, 0..., 1)\}$ and $U_2 = \mathbb{S}^n - \{(0, 0, 0..., -1)\}$. And $\phi_1(x_1, x_2, ..., x_{n+1}) = (\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, ..., \frac{x_n}{1-x_{n+1}})$ and $\phi_2(x_1, x_2, ..., x_{n+1}) = (\frac{x_1}{1+x_{n+1}}, \frac{x_2}{1+x_{n+1}}, ..., \frac{x_n}{1+x_{n+1}})$.

Now, $\phi_1^{-1}(x_1, x_2, ..., x_n) = (\frac{2x_1}{x_1^2+x_2^2+...+x_n^2+1}, ..., \frac{x_1^2+x_2^2+...+x_n^2-1}{x_1^2+x_2^2+...+x_n^2+1})$ and

$\phi_2^{-1}(x_1, x_2, ..., x_n) = (\frac{2x_1}{x_1^2+x_2^2+...+x_n^2+1}, ..., \frac{1-x_1^2-x_2^2-...-x_n^2}{x_1^2+x_2^2+...+x_n^2+1})$.

Now, $\phi_1 \circ \phi_2^{-1}(x_1, x_2, ..., x_n) = (\frac{x_1}{x_1^2+x_2^2+...+x_n^2}, ..., \frac{x_n}{x_1^2+x_2^2+...+x_n^2})$. Now, this one is differentiable function and all partial derivatives exists except for origin. And our domain of transition function does not include origin. Hence we get the transition map to be smooth. Similarly, we can prove $\phi_2 \circ \phi_1^{-1}$ is also smooth in its domain of definition. And hence we get a smooth structure on $\mathbb{S}^n$ as $[\mathcal{A}]$. Hence $(\mathbb{S}^n, [\mathcal{A}])$ is a smooth manfold.

Stereographic Projection For n dimensional sphere:

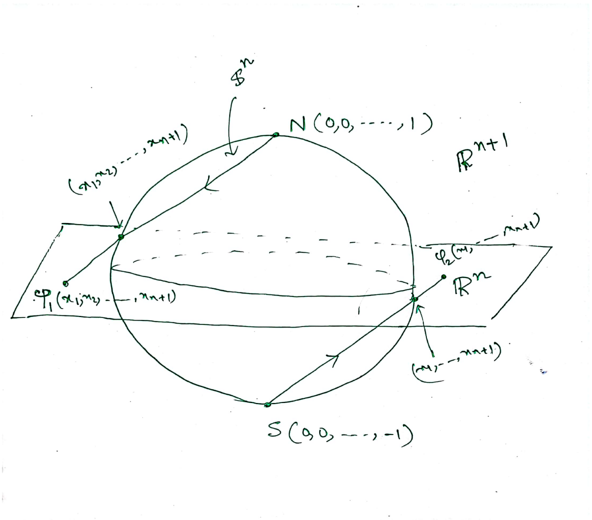


Figure: Stereogrphic Projection to Hyperplane

Differentiability:

Let X_1, X_2 be two smooth manifolds of dim m and n resp. A map $f : X_1 \to X_2$ is called smooth at $p \in X_1$ if given a chart (V, ψ) at $f(p) \in X_2$ there exist a chart (U, ϕ) at $p \in X_1$ such that $f(U) \subseteq V$ and the mapping $\psi \circ \phi^{-1} : \phi(U)(\subseteq \mathbb{R}^m) \to \psi(V)(\subseteq \mathbb{R}^n)$ is smooth at $\phi(p)$. A map f is smooth if it's smootg at every p^t of X_1. The set of all smooth funtios from X_1 to X_2 is denoted by $C^\infty(X_1, X_2)$.
In particular, a map $f : X \to \mathbb{R}$ on a smooth manifold is called smooth if fo all $p \in X$, there exits a chart (U, ϕ) such that the map $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. We denote by $C^\infty(X, \mathbb{R}) = C^\infty(X)$, the set of all real valued smooth functions on X.

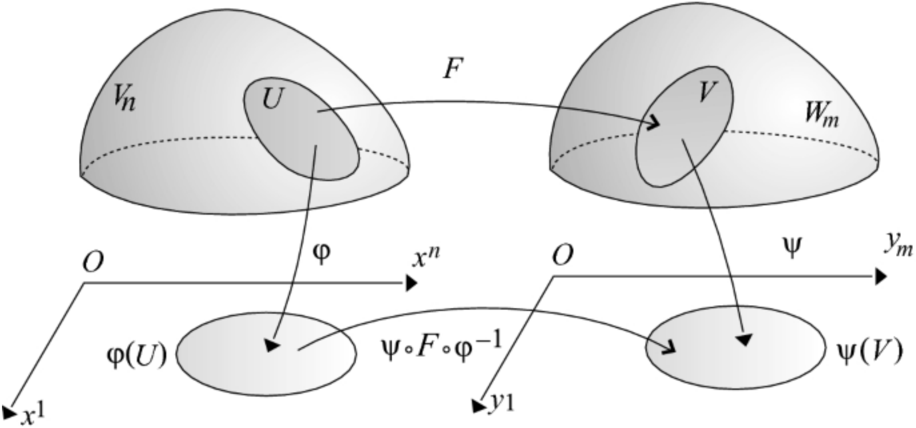


Figure: Defferentiability

1) A diffeomorphism if its bijection and the maps ϕ and ϕ^{-1} are smooth.
2) is a local diffeomorphism at $p \in X$ if there exist neighbourhood U of p and V of $\phi(p)$ such that the map $\phi|_U : U \to V$ is a diffeomorphism .

•**Remark:**While defining differentiability on a smooth manifold we actually use the charts but if we change charts then does it lead to non differentiabilty of the function?

Answer: No as long as we are resctricted ourselves in one differentiable structure! That result leads to the fact that you don't need to worry about analysing differentiabilty usin only one representative atlas from the differentiable structure!

Proof: Let we have two charts (U, ϕ) and (V, ψ) around a point $p \in X$ such that they belong to $\mathcal{A}_1, \mathcal{A}_2$ respectively, and $\mathcal{A}_1, \mathcal{A}_2 \in [\mathcal{A}]$. Then we see that $(f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) = f \circ \psi^{-1}$ on the domain of definition $U \cap V$. Hence we get smoothness and if range set charts change then say we have two compatible charts as far same reasoning viz. $(W_1, \eta_1), (W_2, \eta_2)$ containg $f(p)$, then $(\eta_2 \circ \eta_1^{-1}) \circ (\eta_1 \circ f) = \eta_2 \circ f$. Then this is smooth from the above logic. Whole together we get, $f(U), f(V) \subseteq W_1, W_2$, and hence $f(U \cap V) \subseteq W_1 \cap W_2$. and moreover, we have,

$(\eta_2 \circ \eta_1^{-1}) \circ (\eta_1 \circ f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) = \eta_2 \circ f \circ \psi^{-1}$ telling RHS to be smooth. And Hence as long as e are in fixedd differential stuctres on both domain and range manifolds differetiability can be checked using only one reprsentative atlas taken from each differentiable structures respectively!

Differential of f

: Let $f : X \to \mathbb{R}$ be any smooth function, where X be a smooth manifold.Then we define the following:

•Definition:

- Differential of f: Differential of f t any point $p \in X$ is defined w.r.t. a chart (U, ϕ) about p as differential of $f \circ \phi^{-1}|_{\phi(p)}$. As, we know $f \circ \phi^{-1} : \phi(U) \subseteq (\mathbb{R}^n) \to \mathbb{R}$. Then, it's differential is calculated as normally, $d(f \circ \phi^{-1})|_{\phi(p)} = (\frac{\partial(f \circ \phi^{-1})}{\partial x_1}, \ldots\ldots.., \frac{\partial(f \circ \phi^{-1})}{\partial x_n})|_{\phi(p)}$. And we define defferential of f wrt (U, ϕ) at point p as differential of $f \circ \psi^{-1}$ at point $\phi(p)$ and denoted by $df|_p = d(f \circ \phi^{-1})|_{\phi(p)}$.

Hessian of f

- Hessian of f: The hessian of f w.r.t. (U, ϕ) at point p is defined as hessian of $f \circ \phi^{-1}$ at point $\phi(p)$, and denoted by $H(f)|_p$. Then,

$$H(f)|_p = H(f \circ \phi^{-1})|_{\phi(p)} = J(d(f \circ \phi))|_{\phi(p)}$$

i.e.

$$H(f)|_p = \begin{bmatrix} \frac{\partial^2 F}{\partial^2 x_1^2}|_{\phi(p)} & \frac{\partial^2 F}{\partial x_1 \partial x_2}|_{\phi(p)} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n}|_{\phi(p)} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1}|_{\phi(p)} & \frac{\partial^2 F}{\partial^2 x_2^2}|_{\phi(p)} & \cdots & \frac{\partial^2 F}{\partial x_2 \partial x_n}|_{\phi(p)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1}|_{\phi(p)} & \frac{\partial^2 F}{\partial x_n \partial x_2}|_{\phi(p)} & \cdots & \frac{\partial^2 F}{\partial^2 x_n^2}|_{\phi(p)} \end{bmatrix}_{n \times n}$$

Here, $F = f \circ \phi^{-1}$.

Critical point of f:

- Critical points of f: If p is a critical point or singular point of f if $d(F)|_{\phi(p)}$ is not surjective, this means that the partial derivtives $\frac{\partial F}{\partial x_i}(\phi(p)) = 0$ for all $i = 1, ..., n$.
And the real value $f(p)$ is called the critical point of f.
•**Remark:** Here, all we assume that $\phi(U)$ has dim n and is subset of $\mathbb{R}^n$. We we introduce local coordinates as $\phi(p) = (x_1(p), ..., (x_n(p))$. Now, as ϕ is bijective on U, for on U we can think of $\phi(p)$'s local neighbourhood points as $(x_1, ..., x_n)$ where x_i's are reals such that $||x - \phi(p)|| \leq \varepsilon$ for which $B(\phi(p), \varepsilon) \subseteq \phi(U)$ as there is always a pre-image in U for each such point as ϕ being surjective.

Regular,non-degenerate,degererate,index of non-degenerate points

- Regular Point: Any point which is nit a critical point is called regular point of f and any real value which is not a critical value of f is called regular value of f.
- Non-degenerate Critical Points: $p \in X$ is called non-degenerate critical point of f if the Hessian of f at p is non singular matrix. I.e.

$$det(H(F)|_{\phi(p)}) \neq 0$$

- Degenerate Critical point: Any critical point whose Hessian is singluar is called degenerate critical point.
- Index of a non-degenerate critical point: The index of a non-degenerate critical point is the number of negative eigen values of the Hessian matrix $H(F)|_{\phi(p)}$.

Morse Function:

•Definition: A smooth map on a smooth manifold X, $f : X \to \mathbb{R}$ is a morse function if its all critical points are non-degenerate.
Example of Morse function:A very basic example of morse function is height function on a sphere and torus embedded vertically in $\mathbb{R}^3$.

Example of Morse function

- Let $X = \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. The function $f : X \to \mathbb{R}$ is defined by $(x, y, z) \mapsto z$ is a morse function.
  proof: Let $\phi_1(x_1, x_2, x_3) = (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3})$ and $\phi_2(x_1, x_2, x_3) = (\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3})$ be two charts of $\mathbb{S}^2$. The inverses of ϕ_1, ϕ_2 are given by $\phi_1^{-1}(x_1, x_2) = (\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1})$ and $\phi_2^{-1}(x_1, x_2) = (\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{1-x_1^2-x_2^2}{x_1^2+x_2^2+1})$ respectively.In order to determine the critical points of f, consider the map $f \circ \phi_i^{-1} : \mathbb{R}^2 \to \mathbb{R}$ for each $i = 1, 2$. Note that $(\mathbb{S}^2 - \{S\}, \phi_2)$ is the coordinate chart around $(0, 0, 1)$ and define a map $g = f \circ \phi_2^{-1}$ by $g(x_1, x_2) = \frac{1-x_1^2-x_2^2}{1+x_1^2+x_2^2}$.
  Now, as we see that $\phi_2(\mathbb{S}^2 - \{S\}) = \mathbb{R}^2$ and on whole $\mathbb{R}^2$ g is differentiable and smooth. This neighbourhood works for all point except for south pole i.e.$S = (0, 0, -1)$.
  Similarly we can prove that at point $N = (0, 0, 1)$, f smooth.
  Since, $dg(x_1, x_2) = (\frac{-4x_1}{(1+x_1^2+x_2^2)^2}, \frac{-4x_2}{(1+x_1^2+x_2^2)^2})$,

We have $dg(x_1, x_2) = 0$ iff $x_1, x_2 = 0$.
Hence, $\phi_2^{-1}(0,0) = (0,0,1)$ is the only critical point of f in $\mathbb{S}^2 - \{S\}$. We will find the Hessian of f at point $(0,0,1)$.
$H(g)|_{\phi((0,0,1))} = H(g)|_{(0,0)} = (\frac{\partial^2 g}{\partial x_i \partial x_j}(0,0))_{1 \le i,j \le 2} =$

$$\begin{bmatrix} \frac{-4(1-3x_1^2+x_2^2)}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} & \frac{16x_1x_2}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} \\ \frac{16x_1x_2}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} & \frac{-4(1+x_1^2-3x_2^2)}{(1+x_1^2+x_2^2)^3}\Big|_{(0,0)} \end{bmatrix} =$$

$$\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

This shows that $(0,0,1)$ is a non-degenerate critical point of f with index 2. For, the point $(0,0,-1)$ we use the chart $(\mathbb{S}^2 - \{N\}, \phi_1)$ and similarly shows that $(0,0,-1)$ is the only critical point of f which is non-degenerate with index 0.
•**Remark:** We get on $\mathbb{S}^2$, $(x,y,z) \mapsto z$ is a morse function with only two critical points.

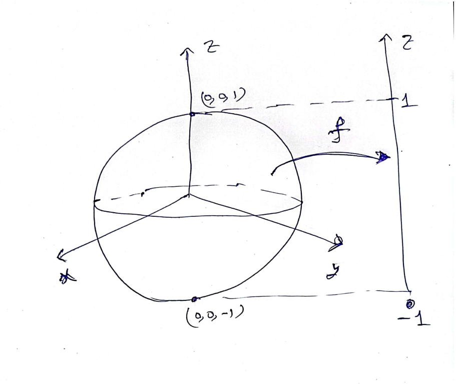


Figure: Morse function on Sphere

•**Question:** Is the converse true?
i.e. If X be a compact, connected, smooth manifold. Suppose there exists a Morse fucntion on X with exactly two critical points. Then M is homeomorphic to a sphere of the corresponding dimension of the manifold.
Answer: Yes! This is proved by mathematician Reeb's.
Finally,

- Let r, R be real numebrs satisfying $0 \leq r \leq R$, and let

$$X = \mathbb{T}^2 = \{(x, y, z) : x^2 + (\sqrt{y^2 + z^2} - R)^2 = r^2\}$$

be a two dimensional torus. Then function $f : \mathbb{T}^2 \to \mathbb{R}$ defined by $f(x, y, z) = z$ is a morse function which has four non-degenerate critical ponits,
$(0, 0, -(R + r)), (0, 0, -(R - r)), (0, 0, R - r), (0, 0, R + r)$.

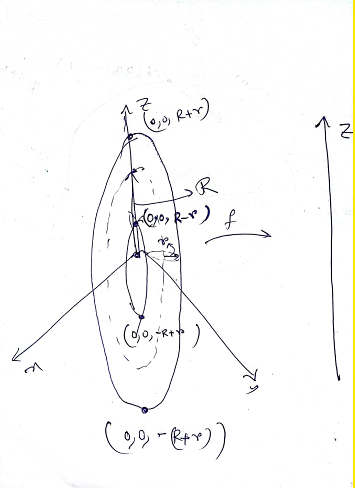


Figure: Morse Function on torus

A standard frame